New bounds for spherical finite distance sets

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Packing problem 1: Independent sets in finite graphs

- In general difficult to solve to optimality (NP-hard)
- The Lovász $\vartheta$-number upper bounds the independence number
- Can be computed through semidefinite programming (SDP)
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A graph whose vertex set is a Hausdorff space is a *topological packing graph* if each finite clique is contained in an open clique.
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![Diagram of spherical caps](image)

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Packing problem 3: Almost finite graphs

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Motivating example: Spherical finite distance graphs
- Two vertices $x, y \in S^{n-1}$ are adjacent if $x \cdot y \not\in \{1, a_1, \ldots, a_r\}$
- $|I/\Gamma| < \infty$ follows from the fact that any two isometric sets in $\mathbb{R}^n$ are related by an isometry of $\mathbb{R}^n$
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Packing problem 3: Almost finite graphs

- Can use bounds for spherical finite distance graphs to obtain bounds on the maximum number of equiangular lines and nonexistence proofs of strongly regular graphs.
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- Can use bounds for spherical finite distance graphs to obtain bounds on the maximum number of equiangular lines and nonexistence proofs of strongly regular graphs
- The Delsarte 2-point bound and Bachoc-Vallentin 3-point bound have been studied extensively in the context of spherical finite distance graphs and equiangular lines [Delsarte, Goethals, Seidel, Barg, Yu, King, Tang, Glazyrin]
A hierarchy of $k$-point bounds for packing problems

- $I_{k-2}$ is the set of independent sets of cardinality $\leq k - 2$
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\[
\Delta_k(G)^* = \inf \left\{ \alpha : \alpha \in \mathbb{R}, T \in \mathcal{C}(V^2 \times I_{k-2}) \succeq_0, B_kT \leq (\alpha - 1)1_{I=1} - 21_{I=2} \right\}
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$$\Delta_k(G)^* = \inf \left\{ \alpha : \alpha \in \mathbb{R}, T \in \mathcal{C}(V^2 \times I_{k-2})_{\geq 0}^\Gamma, B_k T \leq (\alpha - 1)1_{I=1} - 21_{I=2} \right\}$$

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- $T(\gamma x, \gamma y, \gamma Q) = T(x, y, Q)$ for all $\gamma \in \Gamma$
Symmetry reduction

Lemma (L-Machado-Oliveira-Vallentin 2018)
If $I_{k-2}/\Gamma$ is finite, then we have the homeomorphism

$$\bigsqcup_{R \in R_{k-2}} V^2/\text{Stab}_\Gamma(R) \simeq (V^2 \times I_{k-2})/\Gamma,$$

where $R_{k-2}$ a complete set of representatives of the orbits of $I_{k-2}$.
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**Lemma** (L-Machado-Oliveira-Vallentin 2018)
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Corollary
If $I_{k-2}/\Gamma$ is finite, then we have the isomorphism

$$
\Psi: \bigoplus_{R \in \mathcal{R}_{k-2}} C(V^2)^{\text{Stab}_\Gamma(R)} \rightarrow C(V^2 \times I_{k-2})^\Gamma
$$

given by $\Psi(K)(x, y, Q) = K_{\gamma_Q^{-1}Q}(\gamma_Q^{-1}x, \gamma_Q^{-1}y)$
Stabilizer invariant kernels

- Let $R \in \mathcal{R}_{k-2}$ with $k \leq n$; assume vectors in $R$ independent
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**Theorem**  (Musin 2014 / Nonorthogonal extension LMOV 2018)

Every

\[
K \in \mathcal{C}(S^{n-1} \times S^{n-1})^{\text{StabO}(n)}(\text{span}(R))
\]

can be approximated uniformly by kernels of the form

\[
K(x, y) = \sum_{l=0}^{d} \text{trace}(F_l Y_l^{n,m}(x \cdot y, L_{AR} x, L_{AR} y)),
\]

where the matrices \( F_l \) are positive semidefinite
The cardinality of $I_{k-2}/\Gamma$

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- Implementation of $\Delta_k(G)^*$ for finite spherical distance graphs for general $k$.
- Currently computations for $k = 4, 5$. 
Spherical finite distance graph with $a_1 = a$, $a_2 = -a$
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Adaptation of the Lasserre hierarchy for packing

**Definition** (L-Vallentin 2015):

\[
\text{last}_t(G)^* = \inf \left\{ K(\emptyset, \emptyset) : K \in \mathcal{C}(I_t \times I_t)_{\succeq 0}, \right. \\
\left. \quad A_t K(S) \leq -1_{I_{=1}}(S) \text{ for } S \in I_{2t}' \right\}
\]
Adaptation of the Lasserre hierarchy for packing

Definition (L-Vallentin 2015):

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\[ A_t : \mathcal{C}(I_t \times I_t) \to \mathcal{C}(I_{2t}), \quad A_t K(S) = \sum_{J, J' \in I_t : J \cup J' = S} K(J, J') \]
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l_{as_t}(G)^* = \inf \left\{ K(\emptyset, \emptyset) : K \in \mathcal{C}(I_t \times I_t) \geq 0, \right. \\
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Theorem

Convergence: \( l_{as_{\alpha(G)}}(G)^* = \alpha(G) \)
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**Definition** (L-Vallentin 2015):

\[ \text{las}_t(G)^* = \inf \left\{ K(\emptyset, \emptyset) : K \in C(I_t \times I_t) \geq 0, \right. \]
\[ \left. A_t K(S) \leq -1_{I=1}(S) \text{ for } S \in I_{2t}' \right\} \]

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**Theorem**

Convergence: \( \text{las}_{\alpha(G)}(G)^* = \alpha(G) \)

(The proof uses the primal)
Adaptation to energy minimization (L-2016)

The following optimization problem gives a lower bound on the ground state energy of \( N \) particles in \( V \) with pair potential \( f \):

\[
E_t^* = \sup \left\{ \sum_{i=0}^{2t} \binom{N}{i} a_i : a \in \mathbb{R}^{\{0, \ldots, 2t\}}, K \in \mathcal{C}(I_t \times I_t) \succeq 0, a_i + A_t K(S) \leq f(S) \right. \\
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- Finite convergence: \( E_N^* \) is equal to the ground state energy
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$$\quad \left. \text{for } S \in I_{=i} \text{ and } i = 0, \ldots, 2t \right\}$$

- Finite convergence: $E_N^*$ is equal to the ground state energy
- $E_1^*$ is essentially the Yudin bound
Adaptation to energy minimization (L-2016)

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  the Riesz $s$-potentials with $s = 1, \ldots, 7$
- $N = 5$ particularly interesting because of the phase transition
- See Schwartz’ talk on Friday for his approach that solves this
  problem for all $s$ in an interval containing the phase transition
Specialization to finite distance graphs (LMOV 2018)

\[
\text{last}_t(G)^* = \inf \left\{ K(\emptyset, \emptyset) : K \in \mathcal{C}(I_t \times I_t)_{\succeq 0}, \right.
\]

\[
A_t K(S) \leq -1_{I=1}(S) \text{ for } S \in I'_{2t} \left\}
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- May assume \( K \) is \( O(n) \)-invariant
- Again only finitely many linear constraints (one for each orbit)
- Need to describe the cone \( \mathcal{C}(I_t \times I_t) \)
- Fourier inversion: \( K(J,J') = \sum \pi \text{trace}(F\pi Z\pi(J,J')) \)
- Need to compute the zonal matrices \( Z\pi(J,J') \)
Specialization to finite distance graphs (LMOV 2018)

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\lambda_{st}(G)^* = \inf \left\{ K(\emptyset, \emptyset) : K \in C(I_t \times I_t)_{\geq 0}, \right. \\
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Approach via the addition formula

- Decompose into $O(n)$-irreducibles: $\mathcal{C}(I_t) = \bigoplus_{\pi} \bigoplus_{i=1}^{m_{\pi}} H_{\pi,i}$
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- Slow for large $n$

- This is like generating all spherical harmonics if you only need the Jacobi polynomials
Connection to the Stiefel harmonics

- Let $\text{Hom}_{O(n)}(I_t, H_\pi)$ be the space of continuous $O(n)$-equivariant maps $I_t \rightarrow H_\pi$
Connection to the Stiefel harmonics

- Let $\text{Hom}_{O(n)}(I_t, H_\pi)$ be the space of continuous $O(n)$-equivariant maps $I_t \to H_\pi$

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- We have

$$\text{Hom}_{O(n)}(I_t, H_\pi) \simeq \bigoplus_{R \in \mathcal{R}_t} H^\text{Stab}_{O(n)}(R)$$

where $\mathcal{R}_t$ is a complete set of representatives of the orbits
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- Find the right representations $H_\pi$ of $O(n)$
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where $\mathcal{R}_t$ is a complete set of representatives of the orbits
- Find the right representations $H_\pi$ of $O(n)$
- We are essentially interested in

$$H^{SO(n-i)}_\pi \quad \text{for} \quad i = 0, \ldots, t$$

where $\pi$ is a representation of $SO(n)$
Connection to the Stiefel harmonics

- By Frobenius reciprocity we have

\[ \dim(H^{SO(n-t)}_\pi) = \text{mult}(H_\pi, L^2(SO(n)/SO(n-t))) =: m_\pi \]
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[Gelbart 1974] showed \( m_\pi \lambda \) seems to be an act of providence.
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where $p_{\pi,i,i',J,J'}$ is some explicitly computable polynomial
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- The implementation is work in progress
Thank you!